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# THE PERTURBED TIME-OPTIMAL PROBLEM OF CONTROLLING THE FINAL POSITION OF A MATERIAL POINT BY MEANS OF A LIMITED FORCE†

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The problem of the time-optimal incidence on to a desired point of geometrical space of a perturbed dynamical system on which a controlling force of limited modulus acts is investigated. The mathematical apparatus for solving this problem is based on the use of methods of the theory of optimal control in the form of the maximum principle and of a small parameter (of regular perturbations). The generating (unperturbed) problem of optimal control of the motion in both the open-loop and feedback forms is studied in detail. The phenomenon of the irregular dependence of the feedback control and of the Bellman function of the system on the phase variables (geometrical coordinates and velocities) is found and analysed in detail in convenient self-similar variables. An algorithm for solving the problem taking perturbing factors of general form into account is developed and illustrated by an example. Its justification requires further study.

## 1. STATEMENT OF THE PROBLEM AND THE MAXIMUM PRINCIPLE

A perturbed controllable dynamical system described by the second-order vector equation with known initial data

$$\ddot{x} = u + \varepsilon f(x, \dot{x}), \quad x(0) = x^0, \quad \dot{x}(0) = \dot{x}^0 \tag{1.1}$$

$$x \in E^n, \quad n \geqslant 2, \quad \dot{x} = dx/dt, \quad \varepsilon \in [0, \varepsilon_0]$$

is considered.

Here $u$ is a control, of limited modulus, which belongs to the class of piecewise-continuous functions of time $t$, and the numerical parameter $\varepsilon$ specifies the magnitude of perturbations of fairly general form $\varepsilon f$ and is henceforth assumed to be small ($\varepsilon \ll 1$), i.e. the control $u$ is the factor governing the acceleration. For system (1.1) the problem of the time-optimal incidence on to a fixed point $x^f$ from a certain domain $D_x$ of the geometrical space $E^n$ : $x$, $x^f \in D_x \subset E^n$ is formulated. We will assume $x^f = 0$ without loss of generality. The final conditions and the functional take the form

$$x(t_f) = 0, \quad \dot{x}(t_f) \in D_{\dot{x}} \subset E^n; \quad t_f \to \min, \quad |u| \leqslant u_0 \tag{1.2}$$

Here $D_{\dot{x}}$ is an admissible region of variation of the velocity vector $\upsilon = \dot{x}$, $u_0 = \text{const}$. Also without loss of generality, we can fix the quantity $u_0$, in particular, we can put $u_0 = 1$. This

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simplification is achieved by dividing Eq. (1.1) by $u_0$ and introducing the new time $t' = u_0^{1/2} t$ with subsequent rewriting of the expressions for $x$, $\dot{x}$ and $f$.

It is required to construct the optimal control in the open-loop $u_p = u_p^*(t, x^0, \dot{x}^0, \varepsilon)$ or feedback form $u_s = u_s^*(x, \upsilon, \varepsilon)$, the optimal phase trajectory $x = x^*(t, x^0, \dot{x}^0, \varepsilon)$, $\upsilon = \upsilon^*(t, x^0, \dot{x}^0, \varepsilon)$ and the minimum value of the functional $t_f = t_f^*(x^0, \dot{x}^0, \varepsilon)$ and also the Bellman function of the problem $T = T(x, \upsilon, \varepsilon)$.

Note that the investigation of problem (1.1) and (1.2) may be of interest from the theoretical and practical points of view. Earlier [1], the limiting case of the unperturbed problem ($\varepsilon = 0$) was outlined. The use of the methods of perturbation theory [2] is associated with difficulties caused by restrictions on the control and by its non-smoothness with respect to the initial values of the phase variables $x^0$ and $\dot{x}^0$ and the parameter $\varepsilon$. The construction of the perturbed solution requires a detailed study of the generating problem of optimal control, which is extremely non-trivial for $n \geqslant 2$. An analytical investigation for the case of arbitrary dimensions is essentially equivalent to the case $n = 2$ (the plane problem).

We will use the necessary conditions of optimality in the form of the maximum principle [1]. For convenience, we will introduce the phase variable $\upsilon = \dot{x}$ (the velocity) and write out the corresponding two-point boundary-value problem for the Hamiltonian system of the form

$$\dot{x} = \upsilon, \quad \dot{\upsilon} = u^* + \varepsilon f(x, \upsilon), \quad u^* = \eta \equiv q|q|^{-1}, \quad |u^*| = 1 \tag{1.3}$$

$$\dot{p} = -\varepsilon(q, f_x'), \quad \dot{q} = -p - \varepsilon(q, f_\upsilon')$$

$$x(0) = x^0 \neq 0, \ \upsilon(0) = \upsilon^0, \quad x(t_f) = 0, \quad q(t_f) = 0, \quad |p(t_f)| = 1$$

Here $p$ and $q$ are the adjoint variables related to $x$ and $\upsilon$ respectively, $u^*$ is the optimal control, $|u^*| = 1$, $\eta$ is a unit vector, $f_x'$ and $f_\upsilon'$ are square $n \times n$ matrices, and the expressions corresponding to them are understood in the following sense: $(q, f_{x,\upsilon}') = ((q, f)_{x,\upsilon}')^T$ where $(q, f)$ is the scalar product of vectors in $E^n$. For convenience, the vector $p$ can be normalized so that $|p(t_f)| = 1$; this will imply that optimal control $u^*$ (1.3) will be non-singular, that is $q \not\equiv 0$. We shall assume that the this normalization is satisfied. To determine the $4n+1$ unknown parameters (the constants of integration and time $t_f$) we have $4n+1$ conditions, namely, $2n$ initial conditions for $x$ and $\upsilon$, $2n$ final conditions for $x$ and $q$ and the normalization condition for $p^f = p(t_f)$.

Thus, it is required to construct and analyse the solution of the two-point boundary-value problem for all $x^0 \in D_x$, $\upsilon^0 \in D_\upsilon$ and $\varepsilon \in [0, \varepsilon_0]$ for fairly small $\varepsilon_0$ with a specified degree of accuracy in $\varepsilon$, i.e. to find

$$x = x^*(t, x^0, \upsilon^0, \varepsilon), \quad \upsilon = \upsilon^*(t, x^0, \upsilon^0, \varepsilon), \quad q = q^*(t, x^0, \upsilon^0, \varepsilon)$$

$$u_p = \overset{*}{u}_p(t, x^0, \upsilon^0, \varepsilon) = \eta^*(t, x^0, \upsilon^0, \varepsilon), \quad \eta^* = q^*|q^*|^{-1}, \quad t_f = \overset{*}{t}_f(x^0, \upsilon^0, \varepsilon) \tag{1.4}$$

where $t_f^*$ is the minimum permissible time for the process to be completed. It is assumed that the perturbing vector-function $f$ has smoothness properties in the variables $x \in D_x$, $\upsilon \in D_\upsilon$ sufficient for methods of a small parameter (of perturbation theory [2]) to be applicable.

If the open-loop optimal control $u_p^*$, the corresponding phase trajectory $x^*$, $\upsilon^*$ and the optimal time $t_f^*$ are constructed for arbitrary $x^0 \in D_x$, $\upsilon^0 \in D_\upsilon$ then, following the optimality principle of the method of dynamic programming [1], it is possible to determine the optimal control by feedback (synthesis) $u_s^*$ and the Bellman function $T$

$$u_s = \eta^*(0, x, \upsilon, \varepsilon), \quad T = \overset{*}{t}_f(x, \upsilon, \varepsilon) \tag{1.5}$$

To obtain expressions (1.5) the substitution $t \to t - t_0$, where $t_0$ is the initial instant corresponding to the values $x^0$ and $\upsilon^0$, is carried out in $u_p = \eta^*$ (1.4). In $\eta_f^*$ and $t_f^*$ (1.4) the parameters are then assumed to be $t_0 = t$, $x^0 = x$, $\upsilon^0 = \upsilon$, i.e. the initial values of the variables are assumed to be the current values; hence $t - t_0 \to 0$ (ultimately $t \to 0$), $x^0 \to x$ and $\upsilon^0 \to \upsilon$.

This procedure is of a formal nature; further investigations are required to prove it for the problem considered here of constructing the perturbed feedback control and the Bellman function.

We will now construct the approximate (in $\varepsilon$) solution of the problem of open-loop control (1.4) using the methods of the theory of regular perturbations [2]. In particular, we shall consider the generating problem and find the so-called zero approximation to the solution [1].

## 2. THE ANALYTIC SOLUTION OF THE UNPERTURBED PROBLEM OF OPTIMAL CONTROL

As follows from the conditions of optimality (1.3), when $\varepsilon=0$ the momenta (the adjoint variables) are equal: $p=\xi=\text{const}$, $|\xi|=1$; $q=\xi(t_f-t)$. Hence we find that the open-loop optimal control $u_p^*=\eta=\xi$ is constant. Integrating the equations of the phase trajectory $x$, $\upsilon$ (with $\varepsilon=0$) we obtain the elementary expressions $\upsilon=\upsilon^0+\xi t$, $x=x^0+\upsilon^0 t+\frac{1}{2}\xi t^2$. The final condition (1.3) for the vector $x$ implies the relations which enable us to determine the vector $\xi$ and the instant $t_f$ when the control process is completed

$$\xi=-2(x^0+\upsilon^0 t_f)t_f^{-2} \tag{2.1}$$
$$t_f^4=4(x^{02}+2c|x^0||\upsilon^0|t_f+\upsilon^{02}t_f^2), \quad c=(x^0,\upsilon^0)|x^0|^{-1}|\upsilon^0|^{-1}$$

Note that the unit vector $\xi$ is defined uniquely for a fixed value of $t_f>0$. From (2.1) it follows that $t_f>0$ when $x^0\neq 0$; this is also assumed from the formulation of the problem. Moreover, it turns out that the instant $t_f$ depends on only three parameters: the moduli $|x^0|$ and $|\upsilon^0|$ and the parameter $c$, which is the cosine of the angle between the vectors $x^0$ and $\upsilon^0$, $-1\leqslant c\leqslant 1$. When $c=0$, $\pm 1$ Eq. (2.1) of the fourth degree in $t_f$ can be solved in an elementary but not unique (for $c=-1$) way, see below.

We will now solve the equation in question and analyse its roots. The number of parameters can be reduced to two by introducing self-similar variables—the unknown $\chi=t_f|x^0|^{-1/2}$ and the new parameter $h=|\upsilon^0||x^0|^{-1/2}$. As a result, we obtain an equation of the fourth degree of the form

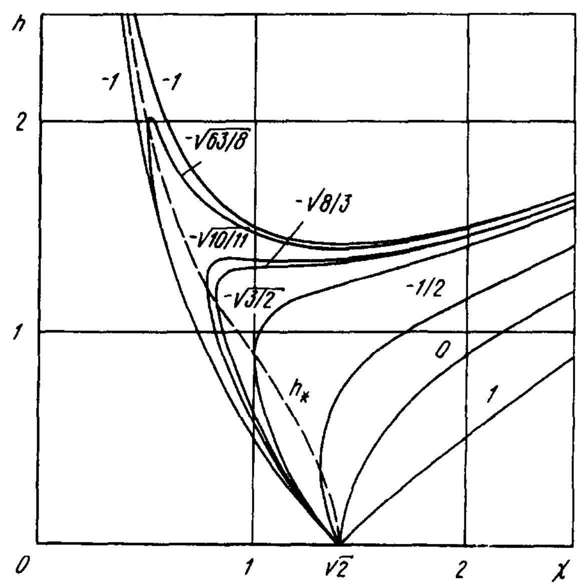


Fig. 1.

$$\chi^4/4 - 1 - 2ch\chi - h^2\chi^2 = 0, \quad c \in [-1, 1], \quad h, \chi \geqslant 0 \tag{2.2}$$

It has positive roots $\chi^*(h, c)$ for all permissible values of $c$ and $h$ (2.2). The solution of Eq. (2.2) can be constructed in graphic form (see Fig. 1) as the one-parameter family of functions. It is natural to take the quantity $h$, $h \in [0, \infty)$ as the argument and the quantity $c$, $c \in [-1, 1]$ as the family parameter. In addition, it turns out that it is preferable to plot graphs of the inverse functions $h = h^*(\chi, c)$. Here $\chi \in (0, \infty)$ is regarded as the argument and $h^*$ is the corresponding root of the quadratic equation. It is convenient to plot $\chi$ along the abscissa axis and $h$ along the ordinate axis (vice versa is also possible which is unimportant). For each solid curve the corresponding values of the family parameter $c$ are shown in Fig. 1.

We will now consider a graphical-analytic procedure for constructing solutions of Eq. (2.2) and, in essence, solve the problem of the optimal control in both the open-loop and feedback forms. By solving (2.2) formally we obtain expressions for $h^*(\chi, c)$ which are real in the domain $(\chi, c) \in \Omega$

$$h = h_{1,2} \equiv \tfrac{1}{2}[-2c\chi \pm d(\chi, c)]\chi^{-2}, \quad d = [\chi^6 - 4\chi^2(1-c^2)]^{1/2} \tag{2.3}$$

$$\Omega = \{\chi, c\colon \chi \geqslant 0, \ -1 \leqslant c \leqslant 1, \ d^2 \geqslant 0\}$$

From (2.3) it follows that $h = h_1(\chi, c)$, $\chi \geqslant \surd(2)$ is the permissible non-negative root for the values $c \in [0, 1]$, where $h_1(\surd(2), c) \equiv 0$, i.e. $t_f^* = \surd(2|x^0|)$ when $\upsilon^0 = 0$, which is obvious. It is not also difficult to establish that the function $h = h_1(\chi, c)$ is approximated by the straight line $h \simeq \chi/2$ (see Fig. 1) as $\chi \to \infty$. The simple asymptotic form $t_f^* \simeq 2|\upsilon^0|$ hence follows, i.e. when moving away from the origin with a high velocity (at the initial instant) half of the time is spent stopping and the second half is spent returning to the terminal manifold $x = 0$. The contribution of $|x^0|$ is relatively small and disappears when $h = |\upsilon^0||x^0|^{-1/2} \to \infty$. As was printed out, if $c = 0$ or $c = 1$, when the vectors $x^0$ and $\upsilon^0$ are orthogonal or collinear, Eq. (2.1) can be solved in an elementary way for $\chi = \chi(h, c)$

$$\chi(h, 0) = \sqrt{2}[h^2 + (h^4 + 1)^{1/2}]^{1/2}, \quad \chi(0, 0) = \sqrt{2} \tag{2.4}$$

$$\chi(h, 1) = h + (h^2 + 2)^{1/2}, \quad \chi(0, 1) = \sqrt{2}$$

$$\chi(h, 0) = 2h + \tfrac{1}{4}h^{-3} + O(h^{-7}), \quad \chi(h, 1) = 2h + h^{-1} + O(h^{-3}), \quad h \to \infty \ (\chi \to \infty)$$

Both curves (2.4) depend on $h$ monotonically and have a similar behaviour when $h \to \infty$ but, when $h \to 0$, we have $\partial\chi(h, 0)/\partial h \to 0$, $\partial\chi(h, 1)\partial h \to 1$. This means that the slope of the tangents is equal to $\pi/2$ and $\pi/4$, respectively, when $c = 0$ and $c = 1$, where $(\partial\chi > 0)\colon h_1(\surd(2) + \partial\chi, 0) = 2^{1/4}\partial\chi^{1/2} + O(\partial\chi^{3/2})$, $h_1(\surd(2) + \partial\chi, 1) = \partial\chi + O(\delta\chi^2)$. Note that the relation $h = h_1(\chi, c)$ is monotone in both variables in the domain $(\chi, c) \in \Omega$ (it is increasing in $\chi$ and decreasing in $c$, $c \in [0, 1]$

$$h_1(\chi'', c) > h_1(\chi', c), \quad (\chi', c) \in \Omega, \quad (\chi'', c) \in \Omega, \quad \chi'' > \chi' \tag{2.5}$$

$$h_1(\chi, c'') > h_1(\chi, c'), \quad (\chi, c') \in \Omega, \quad (\chi, c'') \in \Omega, \quad c'' < c'$$

The relation $\chi = \chi^*(h, c)$ will be monotonically increasing in both variables (see Fig. 1) in the corresponding domain $(h, c) \in \Omega^*$. We have the estimate of difference $\chi^*(h, 1) - \chi^*(h, 0) = h^{-1} + O(h^{-3})$ as $h \to \infty$.

We will now consider expressions (2.3) for negative values of $c$, $-1 \leqslant c < 0$. The permissible set of roots is specified by the relations

$$h = h^*(\chi, c) = \begin{cases} h_1(\chi, c), & \infty > \chi \geqslant \chi_*(c), \quad -1 \leqslant c < 0 \\ h_2(\chi, c), & \sqrt{2} \geqslant \chi \geqslant \chi_*(c), \quad -1 \leqslant c < 0 \end{cases} \tag{2.6}$$

$$\chi_* = \chi_*(c) = \sqrt{2}(1-c^2)^{1/4}, \quad 0 \le \chi_* < \sqrt{2}; \quad \partial h / \partial\chi \to \pm\infty, \quad \chi \to \chi_* \pm 0$$

It follows from (2.6) that the desired root $h$ is determined by two functions $h_{1,2}(\chi, c)$ defined in the domain $(\chi, c) \in \Omega$ by (2.3). Both branches $h_1$ and $h_2$ join smoothly at the points of the line (for fixed values of $c$, $-1 \le c < 0$) in which the discriminant is zero, $d^2 = 0$. The set of these points forms the monotone curve (shown by the dashed line)

$$h = h_*(\chi) = (\chi^{-2} - \chi^2 / 4)^{1/2}, \quad 0 < \chi \le \sqrt{2} \tag{2.7}$$

If the parameter $c$ is fixed, then the joining of the branches $h_1$ and $h_2$ occurs on the curve $h_1(\chi)$ at the point $(\chi_*, h_*)$

$$\chi_* = \sqrt{2}(1-c^2)^{1/4}, \quad h_* = |c|\chi_*^{-1} = |c|/(\sqrt{2}(1-c^2)^{1/4}) \tag{2.8}$$

Again we note that the tangent to the curve $h^*(\chi, c)$ at this point is vertical and the tangent to the curve $\chi^*(h, c)$ is horizontal.

Consider the limiting case $c = -1$ (the vectors $x^0$ and $\upsilon^0$ are anticollinear). For $h = h^*(\chi, -1)$ we have two permissible limit expressions ($\chi_* = \chi_*(-1) = 0$, $h_*(-1) = +\infty$)

$$\begin{aligned} h &= h_1(\chi, -1) = \chi^{-1} + \chi/2, \quad \infty > \chi > 0 \\ h &= h_2(\chi, -1) = \chi^{-1} - \chi/2, \quad 0 < \chi \le \sqrt{2} \end{aligned} \tag{2.9}$$

These curves "bound" the families of curves $h_{1,2}(\chi, c)$ as $c \to -1$; the qualitative analysis of them is fairly elementary. It is important to note that $h_{1,2}(\chi, -1) \to \infty$ as $\chi \to 0$, moreover $h_1 > h_2$ ($\sqrt{(2)} \ge \chi > 0$). Furthermore, we have $h_2(\sqrt{(2)}, -1) = 0$ while $h_1(\sqrt{(2)}, -1) = \sqrt{(2)}$ is the minimum value. In addition we note that $h_1(\chi, -1) - h_2(\chi, -1) = \chi \to 0$ as $\chi \to 0$, $h_1(\chi, -1) - h_1(\chi, 1) \to 0$ (from above) as $\chi \to \infty$, and $h_1(\sqrt{(2)}, 1) = h_2(\sqrt{(2)}, -1)$. The branches $h = h_{1,2}(\chi, -1)$ are solved in an elementary way for the unknown $\chi$, and non-uniquely for $h = h_1$ (see Fig. 1).

Thus, we have established that the family of curves $h^*(\chi, c)$ is enclosed in the "infinite curvilinear triangle" bounded by the curves $h_1(\chi, \pm 1)$ and $h_2(\chi, -1)$ on the corresponding sets of values of $\chi$. An analogous assertion holds for the family of curves $\chi^*(h, c)$. The curve of "joining" $h = h_*(\chi)$ (2.7) is inside the "triangle" and strictly between the curves $h_{1,2}(\chi, -1)$; more precisely, $h_1(\chi, -1) > h_*(\chi) > h_2(\chi, -1)$, $0 < \chi \le \sqrt{(2)}$.

We will now continue to analyse the "fine structure" of the family of curves $h^*(\chi, c)$ (2.6). The behaviour of the bounding curves $h_{1,2}(\chi, -1)$, $0 < \chi \le \sqrt{(2)}$ and the joining curve $h_*(\chi)$ implies that, as $c$ becomes smaller, the dependence of $\chi = \chi^*(h, c)$, $c_* > c \ge -1$ on $h$ will be non-unique beginning with a certain value $c_*$, $0 > c_* > -1$. This means that points of local extrema, for which the condition $\partial h_1 / \partial\chi = 0$ holds, appear on the curve $h_1(\chi, c)$, $c \le c_*$, $\chi > \chi_*$, see (2.8). Solving this equation for $\chi$ we find the desired quantity $c_*$ and also the extremum points. It turns out there are only two such points: the point of maximum and the point of minimum. Thus, we have (see Fig. 1)

$$\begin{aligned} &c_* = -\sqrt{8}/3 \simeq -1 + 1/18, \quad -1 \le c \le c_* \\ &\chi_{\max,\min}(c) = [6c^2 - 4 \mp 6|c|(c^2 - 8/9)^{1/2}]^{1/4} > \chi_*(c) \end{aligned} \tag{2.10}$$

As $c \to -1$, the point $\chi_{\max} \to 0$ and the point $\chi_{\min} \to \sqrt{2}$; here $h_{\max}(c) = h_1(\chi_{\max}(c), c) \to \infty$, and $h_{\min}(c) = h_1(\chi_{\min}(c), c) \to \sqrt{2}$ is the minimum value of $h_1(\chi, -1)$ (see above and Fig. 1). It is interesting to estimate the excess of $h_{\max}(c)$ over the quantity $h(\chi_*(c), c)$ corresponding to the joining point (2.8). We have

$$\begin{aligned} &\chi_{\max}(c) = (8\delta c)^{1/4}(1 + (15/8)\delta c + O(\delta c^2)), \quad c = -1 + \delta c \\ &h_1(\chi_{\max}(c), c) = (8\delta c)^{-1/4}(1 + (9/8)\delta c + O(\delta c^2)), \quad \delta c > 0 \end{aligned}$$

$$h^*(\chi_*(c),c) = (8\delta c)^{-1/4}(1-(7/8)\delta c + O(\delta c^2)) \tag{2.11}$$

$$h_1(\chi_{\max}(c),c) - h^*(\chi_*(c),c) = (2\delta c^3)^{1/4} + O(\delta c^{7/4})$$

By (2.11) this excess is insignificant and tends to zero quite rapidly as $\delta c \to 0$.

The above analysis of the function $h = h^*(\chi, c)$ (2.3), (2.6) provides an algorithm for constructing the solution of the unperturbed problem of the optimal open-loop control according to (2.1). Suppose the initial vectors $x^0$ and $\upsilon^0$ are known. Using these the quantities $h = |\upsilon^0|\,|x^0|^{-1/2}$ and $c = (x^0, \upsilon^0)|x^0|^{-1}|\upsilon^0|^{-1}$ are constructed and the minimum root $\chi = \chi^*(h, c)$ of Eq. (2.2) is found by a numerical or graphical-analytical method (see Fig. 1). For $0 \leqslant c \leqslant 1$ the dependence is one-to-one and monotonically increasing both in the argument $h$ and in the family parameter $c$. The function $\chi = \chi^*(h, c)$ is single-valued and smooth in the range of variation of the parameter $c$, $0 \geqslant c > c_* = -\sqrt{(8/3)}$ but is not one-to-one: the same values of $\chi$ may correspond to two different values of the argument $h$, i.e. the function $h = h^*(\chi, c)$ is two-to-one in a certain range of variation of $\chi$. Thus, for values $1 \geqslant c > c_* = -\sqrt{(8/3)}$ the smooth single-valued dependence $\chi = \chi^*(h, c)$ occurs for all $h \geqslant 0$ and hence a smooth single-valued solution of the optimal control problem with respect to the initial data exists.

The situation changes in a qualitative and important way for the values $c$ in the range $-1 \leqslant c \leqslant c_* = -\sqrt{(8/3)}$. If the value of $h$ is "not too large", i.e. $0 \leqslant h < h_{\max}(c)$, then the dependence of the minimum root $\chi = \chi^*(h, c)$ is single-valued and smooth. When the argument $h$ passes (increasing or decreasing) through the value ($h \gtrless h_{\max}$), a sudden change in the quantity $\chi = \chi^*(h, c)$ by a finite significant amount $\pm\Delta\chi$ occurs

$$\Delta\chi = \chi^*(h_{\max}(c)+0, c) - \chi_{\max}(c) = 2h_{\max}(c)\,(1 - h_{\max}^{-2} + O(h_{\max}^{-3})), \quad h_{\max} \to \infty \tag{2.12}$$

As $c \to -1$ the difference (2.12) increases without limit and, by (2.11), we have the estimate $\Delta\chi \simeq 2(8\delta c)^{-1/4}$, $\delta c \to 0$ for the jump. A further increase in the parameter $h$ ($h > h_{\max}(c)$) again leads to a single-valued monotone dependence, analogous to the case $0 \leqslant c \leqslant 1$. Note that the range of values of the parameter $c$, $c_* \geqslant c \geqslant -1$, in which the irregularity mentioned above occurs, is extremely narrow $c_* + 1 = -\sqrt{(8/3)} + 1 \simeq 1/18$, i.e. the vectors $x^0$ and $\upsilon^0$ are practically anticollinear. The sine of the angle $s$ between them is small: $|s| \leqslant 1/3$.

Thus, the fairly dense family of curves $h = h^*(\chi, c)$ enables us to solve the unperturbed problem of the time-optimal "hard encounter" both by the open-loop (1.4), (2.1) and feedback (1.5) control ($\varepsilon = 0$). The feedback control requires fairly accurate measurement of the phase vector $(x, \upsilon)$ at any current instant $t$. The optimal controlling vector $u_s$, i.e. the unit vector $\xi$, is also determined at each instant of time

$$u_s = \xi^*(x, \upsilon) = -2(x + \upsilon T)T^{-2}, \quad T = t_f^*(x, \upsilon) \tag{2.13}$$

The Bellman function of the problem $T = T(x, \upsilon)$ is the time interval "preceding the encounter" when motion from the admissible position $(t, x, \upsilon)$ occurs.

*Notes.* 1. A somewhat more general case of restrictions on a control can be reduced to the problem investigated above. Suppose we have a controllable system of the form $\dot{x} = \upsilon$, $\dot{\upsilon} = u$, $u = \Lambda U$, where $|U| \leqslant 1$, and $\Lambda$ is a non-degenerate $n \times n$ matrix, i.e. the set of controls is the non-degenerate ellipsoid $(u\Lambda^{-1}, u\Lambda^{-1}) \leqslant 1$. By the non-singular substitution $x = \Lambda X$, $\upsilon = \Lambda V$ this system can be reduced to the form of the system examined above.

2. The problem of the "hard encounter" can be reduced to solving and analysing the roots of an equation of the fourth degree, similar to (2.1) or (2.2) to a constant acceleration $\ddot{x} = u + w$, $w = \text{const}$, $|w| = w_0 < 1$. In this case the number of parameters increases considerably (by three)

$$(1 - w_0^2)\chi^4/4 = 1 + 2ch\chi + h^2\chi^2 + w_0\chi^2(c_x + c_\upsilon h\chi) \tag{2.14}$$

$$c_x = (ww_0^{-1}, x^0|x^0|^{-1}), \quad c_\upsilon = (ww_0^{-1}, \upsilon^0|\upsilon^0|^{-1})$$

When $w_0 \ll 1$ the methods of perturbation theory can be applied to (2.14), (see Section 4). Near the value $h_{max}(c)$ the expansions of $\chi$ are carried out in fractional powers of the parameter $w_0$ (in powers of $w_0^{1/2}$).

3. Suppose we consider $n \geq 2$ controllable systems with one degree of freedom: $\ddot{x} = u_i$, $|u_i| \leq u_{i0}$, $i = 1, \ldots, n$. We will solve the problem of time-optimal incidence on the axes $x_i(t_i) = 0$, where $\dot{x}_i(t_i)$ is arbitrary. We have the controls $u_i = -u_{i0}\mathrm{sign}x_i^0$ and the quadratic equations to find the instants $t_i : x_i^0 + \upsilon_i t_i + \frac{1}{2} u_i t_i^2 = 0$, $i = 1, \ldots, n$. Let us require that these conditions are to be satisfied at the same instant of time $t_i = t_f$. We then obtain the desired values of the controls $u_i = -2(x_i^0 + \upsilon_i^0 tf) tf^{-2}$. If the condition $u_1^2 + \ldots + u_n^2 = 1$ is imposed, then an equation of the fourth degree, which reduces to (2.2), is obtained for finding the unknown quantity. By choosing the minimum root $t_f^*$ we find the controls $u_i$ which reduce the variables $x_i$ to zero simultaneously in the same minimum period of time. When using this approach, the quantities $u_{i0}$ may have no meaning; the quantities $u_i$ certainly have the meaningful sense as the components of the control vector $u$ for an n-dimensional dynamical system.

4. It is interesting to note that some relations analogous to those obtained above in Section 2 also hold in the problem of minimizing misses [3].

## 3. AN ALGORITHM FOR THE APPROXIMATE SOLUTION OF THE PERTURBED PROBLEM

Let us consider the perturbed boundary-value problem of the maximum principle (1.3). In the first stage of discussions we shall assume that the optimal time $t_f$ is given and, using the methods of perturbation theory, we shall find the variables $x, \upsilon, p, q$ with the desired degree of accuracy in $\varepsilon$ as the solution of the system of $4n$ integral equations

$$\begin{aligned}
&p = p^f + \varepsilon P, \quad q = p^f(t_f - t) + \varepsilon Q, \quad u^* = q|q|^{-1} \equiv \eta^f + \varepsilon U \\
&\eta^f = p^f|p^f|^{-1}, \quad \varepsilon U \equiv (\eta^f + \varepsilon N)|\eta^f + \varepsilon N|^{-1} - \eta_f, \quad N = Q|p^f|^{-1}(t_f - t)^{-1} \\
&\upsilon = \upsilon^0 + \eta^f t + \varepsilon V, \quad x = x^0 + \upsilon^0 t + {}^1/_2 \eta^f t^2 + \varepsilon X \\
&P \equiv -\int_{t_f}^{t} (q, f_x')d\tau, \quad Q \equiv -\int_{t_f}^{t} [P + (q, f_\upsilon')]d\tau \\
&V \equiv \int_0^t (U + f)d\tau, \quad X \equiv \int_0^t V d\tau = \int_0^t (t - \tau)(U + f)d\tau
\end{aligned} \tag{3.1}$$

Here we have introduced the parameter $\varepsilon$ and the non-linear integral operators $P, Q, U, V, X$ of the variables $q, \upsilon, x$ containing the unknowns $p^f$, $t_f$ ($t_f$ is considered to be given for now). Without loss of generality and in accordance with the maximum principle we can put $|p^f| = 1$ in (3.1); then $\eta^f = p^f$ is the constant unit vector to be determined from the final condition for $x$.

The functions $q, \upsilon, x$ (the vector $p$ is not needed) are constructed by successive approximations in powers of $\varepsilon$ (Picard's scheme) or by Taylor-series expansions (in the case when the function $f$ is sufficiently smooth in $x$ and $\upsilon$). The recurrent scheme for determining the unit vector $\eta^f$ and the control $u^*$ may be realized simultaneously with the one mentioned. As a result we have the algorithm

$$\begin{aligned}
&p_i = \eta_i^f + \varepsilon P_{i-1}, \quad q_i = \eta_i^f(t_f - t) + \varepsilon Q_{i-1}, \quad u_i^* = \eta_i^f + \varepsilon U_{i-1} \\
&\upsilon_i = \upsilon^0 + \eta_i^f t + \varepsilon V_{i-1}, \quad x_i = x^0 + \upsilon^0 t + \tfrac{1}{2}\eta_i^f t^2 + \varepsilon X_{i-1} \\
&\eta_i^f = -2t_f^{-2}(x^0 + \upsilon^0 t_f + \varepsilon X_{i-1}^f), \quad X_i^f \equiv \int_0^{t_f} (t_f - t)(U_i + f_i)dt
\end{aligned} \tag{3.2}$$

$$i=1,2,\ldots,;\quad p_0=\eta_0^f,\quad q_0=\eta_0^f(t_f-t),\quad u_0^*=\eta_0^f$$
$$\upsilon_0=\upsilon^0+\eta_0^f t,\quad x_0=x^0+\upsilon^0 t+\tfrac{1}{2}\eta_0^f t^2,\quad \eta_0^f=-2t_f^{-2}(x^0+\upsilon^0 t_f)$$

The operators $P_{i-1}$, $Q_{i-1}$, $U_{i-1}$, $V_{i-1}$, $X_{i-1}$ are defined in terms of $q_{i-1}$, $\upsilon_{i-1}$, $x_{i-1}$ and $h_{i-1}^f$, i.e. they are known functions of the parameters $x^0$, $\upsilon^0$, $\varepsilon$ and $t_f$. By the theorem on the contraction operator [4], when $\varepsilon>0$ is sufficiently small, the successive approximations converge uniformly in a certain domain $x^0\in D_x$, $\upsilon^0\in D_\upsilon$, $0<t_*\leqslant t_f\leqslant t^*<\infty$ and determine the relation for finding $t_f$

$$\begin{aligned}&\chi^4/4-1-2ch\chi-h^2\chi^2=\varepsilon\kappa(\chi,x^0,\upsilon^0,\varepsilon)\\&\kappa(\chi,x^0,\upsilon^0,\varepsilon)\equiv 2|x^0|^{-1}(c_x+c_\upsilon h\chi)|X^f|\\&c_x=(x^0|x^0|^{-1},\ X^f|X^f|^{-1}),\quad c_\upsilon=(\upsilon^0|\upsilon^0|^{-1},\ X^f|X^f|^{-1})\end{aligned}\tag{3.3}$$

Here $X^f=X^f(\chi,\ x^0,\ \upsilon^0,\ \varepsilon)$ is a known function which is constructed by the scheme (3.2) with the prescribed range of accuracy. It is sufficiently smooth with respect to the parameter $\chi$ over a certain range of variation of $\chi$ which can be as large as desired when $\varepsilon\to 0$. For the purpose of reducing the number of parameters in the generating equation ($\varepsilon=0$) the unknown $\chi=t_f|x^0|^{-1/2}$ and the parameters $h=|\upsilon^0||x^0|^{-1/2}$, $c=(x^0|x^0|^{-1},\ \upsilon^0|\upsilon^0|^{-1})$ specified by measurements have been introduced by dividing by $(x^0)^2$ in (3.3). The coefficients $c_x$ and $c_\upsilon$, i.e. $k$, take into account the perturbation $f$ in system (1.1); they are computed by means of the recurrent scheme (3.2). Note that the representation of the equation specifying $t_f$ in the form (3.3) may turn out to be inconvenient because of the presence of $|x^0|^{-1}$, $|\upsilon^0|^{-1}$, $|X^f|^{-1}$. It is then preferable to use the representation in the form of perturbed equation (2.1)

$$t_f^4/4-x^{02}-2(x^0,\upsilon^0)t_f-\upsilon^{02}t_f^2=2\varepsilon(x^0+\upsilon^0 t_f,X^f)\tag{3.4}$$

It is required to construct the solution of the problem of the optimal control, i.e. to find the minimum root $t_f^*=\min\{t_f\}$ of Eq. (3.4) or (3.3). Everywhere, outside of a small vicinity of the point $\chi_{\max}(c)$, $h_{\max}(c)$, $c\leqslant c_*$, the dependence of $t_f^*$ on $\varepsilon$ will be smooth and the quantity $t_f^*$ can be found in an analytic way to the desired degree of accuracy in $\varepsilon$ by means of the recurrent scheme of the method of successive approximations [5]. The function $t_f^*$ is irregular (it has a discontinuity of the first kind in the variables $\varepsilon$, $x^0$, $\upsilon^0$) near the singular point. The use of more precise numerical methods or more precise analytic constructions of the function $h=h^*(\chi,\ c,\ x^0,\ \upsilon^0,\ \varepsilon)$ and the determination of its maximum in $\chi$ are required here. It is possible to realize these procedures in an algorithmic way on a computer. Thus, the approximate solution (with a given degree of accuracy in $\varepsilon$) of the perturbed problem of the time-optimal "hard encounter" has been reduced to the determination of the minimum root $t_f^*(x^0,\ \upsilon^0,\ \varepsilon)$ of Eq. (3.4) (or (3.3)).

## 4. EXAMPLE

Let us consider the special case of a perturbing function $f$, linear in $x$ and $\upsilon$

$$f(x,\upsilon)=W+Lx+K\upsilon;\quad W,L,K=\text{const}\tag{4.1}$$

Here $L$ and $W$ are arbitrary $n\times n$ matrices and $W$ is an $n$-vector. By the scheme of Section 3, we will find the solution of the time-optimal control problem to a first approximation in $\varepsilon$. By (3.2) we have the expressions (with an error of $O(\varepsilon^2)$)

$$\begin{aligned}&p_{(1)}=\eta^f+\tfrac{1}{2}\varepsilon L^T\eta^f(t_f-t),\quad q_{(1)}=(I+\varepsilon S)\eta^f(t_f-t)\\&S=\tfrac{1}{2}[K^T+\tfrac{1}{3}L^T(t_f-t)](t_f-t)\equiv S_0+S_1t+S_2t^2\end{aligned}\tag{4.2}$$

$$u_{(1)} = q_{(1)}|q_{(1)}|^{-1} = (I+\varepsilon R)\eta^f, \quad R = S + (\eta^f, S\eta^f)I$$
$$\upsilon_{(1)} = \upsilon^0 + \eta^f t + \varepsilon t(\alpha_0 + \tfrac{1}{2}\alpha_1 t + \tfrac{1}{3}\alpha_2 t^2) \equiv \upsilon_0 + \varepsilon V(t, t_f, \eta^f, x^0, \upsilon^0)$$
$$x_{(1)} = x^0 + \upsilon^0 t + \tfrac{1}{2}\eta^f t^2 + \tfrac{1}{2}\varepsilon t^2(\alpha_0 + \tfrac{1}{3}\alpha_1 t + \tfrac{1}{6}\alpha_2 t^2) \equiv x_0 + \varepsilon X(t, t_f, \eta^f, x^0, \upsilon^0)$$
$$\alpha_0 = R_0\eta^f + W + Lx^0 + K\upsilon^0, \quad \alpha_1 = (R_1 + K)\eta^f + L\upsilon^0, \quad \alpha_2 = (R_2 + \tfrac{1}{2}L)\eta^f$$
$$\alpha_i = \alpha_i(t_f, \eta^f, x^0, \upsilon^0), \quad i = 0,1,2$$

Here $I$ is the identity matrix, the matrices $R_i$ $(i = 0, 1, 2)$ are obtained by expanding $R$ in powers of $t$: $R = R_0 + R_1 t + R_2 t^2$, i.e. by substituting the matrix coefficients $S_i$, which depend on the unknown $t_f$, for the matrix $S$. By equating $x_{(1)}(t_f) = 0$ we obtain the desired formula for the unit vector $\eta^f$ to a first approximation in $\varepsilon$

$$\eta^f_{(1)} = \xi - 2\varepsilon t_f^{-2} X_0^f, \quad \xi = -2t_f^{-2}(x^0 + \upsilon^0 t_f)$$
$$X_0^f \equiv X(t_f, t_f, \xi, x^0, \upsilon^0) \tag{4.3}$$

Here $X_0^f$ depends on $t_f$ and the initial values $x^0$ and $\upsilon^0$. From the condition $|\eta^f_{(1)}| = 1$ we obtain an equation of the form (3.4) defining the unknown $t_f^*$

$$t_f^4/4 - x^{02} - 2(x^0, \upsilon^0)t_f - \upsilon^{02}t_f^2 = \varepsilon\Theta(t_f, x^0, \upsilon^0) \tag{4.4}$$
$$\Theta = t_f^2(x^0 + \upsilon^0 t_f, \alpha_0 + \tfrac{1}{3}\alpha_1 t_f + \tfrac{1}{6}\alpha_2 t_f^2)|_{\eta^f = \xi} \equiv 2(x^0 + \upsilon^0 t_f, X_0^f)$$
$$t_f^* = \min\{t_f\}, \quad t_f^* = t_f^*(x^0, \upsilon^0, \varepsilon)$$

Let $t_{f0}^*$ be the simple minimum root of Eq. (4.4) when $\varepsilon = 0$, constructed as in Section 2. We then obtain

$$t_{f(1)}^* = t_{f0}^* + \varepsilon\Theta(t_{f0}^*, x^0, \upsilon^0)[t_{f0}^{*3} - 2(x^0, \upsilon^0) - 2\upsilon^{02}t_{f0}^*]^{-1} \tag{4.5}$$

to a first approximation in $\varepsilon$.

In the case of a simple root, the expression in square brackets is non-zero. At the point of a local maximum of the function $h = h^*(\chi, c)$ (see (2.10) and (2.11)) the root $t_{f0}^*$ is of double multiplicity and the expression in square brackets in (4.5) vanishes. Hence the expansions must be carried out in powers of $\sqrt{\varepsilon}$

$$t_{f(1)}^* = t_{f0}^* + \sqrt{\varepsilon}t_{f1}^* + \varepsilon t_{f2}^* + O(\varepsilon^{3/2})$$
$$t_{f0}^* = t_{f0}^*(|x^0|, |\upsilon^0|, c), \quad t_{f1}^* = -[2\Theta_0/(3t_{f0}^* - 2\upsilon^{02})]^{1/2} \tag{4.6}$$
$$t_{f2}^* = [(\Theta'_{t_f})_0 - t_{f0}^* t_{f1}^{*2}](3t_{f0}^* - 2\upsilon^{02})^{-1}, \quad \Theta_0 = \Theta(t_{f0}^*, x^0, \upsilon^0)$$

Formulae (4.3)–(4.6) of the first approximation are also valid in the case of an arbitrary perturbing function $f$. If $f$ is linear then $X^f$ has form (4.2)–4.4). Note that expression (4.6) for $t_{f1}^*$ must be real.

Let $L = K = 0$, i.e. $f = W = \text{const}$. From (4.2) it then follows that the matrices $R = S \equiv 0$, the open-loop control $u_{(1)} = \eta^f = \text{const}$, the vector $\alpha_0 = W$ and $\alpha_{1,2} = 0$; the function $X = \tfrac{1}{2}Wt^2$. Equation (4.4) for evaluating $t_f$ reduces to form (2.14); $\Theta = t_f^2(x^0 + \upsilon^0 t_f, W)$. Note that if the perturbing function $f$ possesses spherical symmetry in $x$ and $\upsilon$, i.e. $L = lI$, $K = kI$ ($l$ and $k$ are scalars and $I$ is the identity matrix), then the control is $u_{(1)} = \eta^f$, similar to the case $L = K = 0$ since $R \equiv 0$ in consequence of the identity $S \equiv (\eta^f, S\eta^f)I$. The vectors $\alpha_i$ in function (4.2) are given by

$$\alpha_0 = W + lx^0 + k\upsilon^0, \quad \alpha_1 = k\eta^f + l\upsilon^0, \quad \alpha_2 = {}^1/_2 l\eta^f \tag{4.7}$$

As has been proved, for the first approximation in $\varepsilon$, the expression $\eta^f = \xi = -2t_f^{-2}(x^0 + \upsilon^0 t_f)$ must replace $\eta^f$ in (4.7). The further computation of the unknown $t_f^*$ is carried out by the above-mentioned scheme.

In conclusion we note that the analogous procedure of the approximate solution of the problem of "hard encounter" (in the variable $x$) can be generalized for a time-dependent system $f = f(t, x, \upsilon, u)$. The basic method turns out to be suitable in the case of systems with slowly varying parameters and others, for instance, of the form $\ddot{x} = -k\dot{x} + u + \varepsilon f$. In this case the procedure for maximizing the function $(q, u + \varepsilon f)$, $|u| \leq 1$, see [2], is required.

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## REFERENCES

1. PONTRYAGIN L. S., BOLTYANSKII V. G., GAMKRELIDZE R. V. and MISCHENKO Ye. F., *The Mathematical Theory of Optimal Processes*. Nauka, Moscow, 1969.
2. AKULENKO L. D., *Asymptotic Methods of Optimal Control*. Nauka, Moscow, 1987.
3. KUZNETSOV A. G. and CHERNOUS'KO F. L., An optimal control which minimizes the extremum of a function of phase coordinates. *Kibernetika* 3, 50–55, 1968.
4. VULIKH B. Z., *Introduction to Functional Analysis*. Nauka, Moscow, 1967.
5. KOLLATZ L., *Functional Analysis and Computational Mathematics*. Mir, Moscow, 1969.

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